Associated graphs of le-modules

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Let M be an le-module over a commutative ring with unity. In this paper, an associated graph $G(M)$ of M with all nonzero proper submodule elements of M as vertices has been introduced and studied. Any two distinct vertices n and m are adjacent if $n+m=e$. Some algebraic, topological and, graph theoretic properties of le-modules have been established. Also, it is shown that the Beck's conjecture is true for coatomic le-modules.

Keywords: submodule elements, superfluous element, complete graph, connected graph

1. INTRODUCTION:

A. K. Bhuniya and M. Kumbhakar [(Bhuniya and Kumbhakar, 2018, 2019)] introduced and studied a new algebraic structure, which is called an le-modules. An le-module M over a commutative ring R is a complete lattice ordered monoid $(M, +, \leqslant, e)$ with greatest element e and module like action of R on it. A. K. Bhuniya and M. Kumbhakar motivated to study abstract submodule theory from the study of abstract ideal theory, in particular multiplicative lattices and lattice modules. For more details about multiplicative lattices and lattice modules one may refer [(Narayan Phadatare and Kharat, 2019)], [(Ballal and Kharat, 2015)]. The notion of a graph of zero-divisors of a commutative ring was introduced in [(Beck, 1988)] , by studying the coloring of a graph constructed by all elements of a commutative ring R. In [(A. Abbasi, 2015)] A. Abbasi, H. Roshan-Shekalgourabi, D. Hassanzadeh-Lelekaami introduced and studied associated graph on modules over commutative rings. Elham Mehdi-Nezhad and Amir M. Rahimi studied similar type of graph and proved some new results on it. Narayan Phadatare, Sachin Ballal and Vilas Kharat studied graph on multiplication lattice modules by using a non-small element. Also they have introduced semi-complement graph on lattice modules.

In this paper we have introduced and studied associated graph on le-module. We have obtained analogous results as in paper [(A. Abbasi, 2015)] and for it we got some new algebraic results for le-modules.

1.1 Definition:

[(Bhuniya and Kumbhakar, 2018, 2019)] An le-semigroup $(M, +, \leqslant, e)$ is a commutative monoid with the zero element 0_M and is a complete lattice with the greatest element e, that satisfies $m + (\vee_{i\in I} m_i) = \vee_{i\in I}(m + m_i)$. Let $(M, +, \leqslant)$ be an le-semigroup with the zero element 0_M and R be a commutative ring with unity 1_R. Then M is called an le-module over R if there is a mapping $R \times M \to M$ satisfying:

(1) $r(m_1 + m_2) = rm_1 + rm_2$
(2) $(r_1 + r_2)m \leqslant r_1m + r_2m$
(3) $(r_1r_2)m = r_1(r_2m)$
(4) $1_Rm = m$; $0_Rm = r0_M = 0_M$

(5) $r(\vee_{i\in I} m_i) = \vee_{i\in I}(rm_i)$ holds for all $r, r_1, r_2 \in R$, $m, m_1, m_2 \in M$ and $i \in I$ (I is an indexed set).

A ***graph*** G is defined as the pair $(V(G); E(G))$, where $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G. For two distinct vertices n and m, $n - m$ means that n and m are ***adjacent***. The ***degree*** of a vertex n of graph G which is denoted by $deg(n)$ is the number of edges incident on n. If $|V(G)| \geqslant 2$, a ***path*** from n to m is a series of adjacent vertices $n - v_1 - v_2 - \ldots - v_n - m$. In a graph G, the ***distance*** between two distinct vertices n and m, denoted by $d(n;m)$ is the length of the shortest path connecting n and m. The ***diameter*** of a graph G is defined as $diam(G) = sup\{d(n;m)|n, m \in V(G)\}$. A graph G is called ***connected***, if for any vertices n and m of G there is a path between n and m. The ***girth*** of G, is the length of the shortest cycle in G and it is denoted by $g(G)$. A graph is called ***complete*** if each pair of vertices is adjacent. A complete graph with $n-$vertices is denoted by K_n. An $r-partite$ graph is one whose vertex set can be partitioned into r subsets such that no edge has both ends in any one subset. A ***complete*** $r-partite$ graph is one each vertex is joined to every vertex that is not in the same subset. The complete bipartite (i.e, 2-partite) graph with part sizes m and n is denoted by $K_{m,n}$. A ***clique*** of a graph is its maximal complete subgraph and the number of vertices in the largest clique of a graph G, denoted by $\omega(G)$, is called the ***clique number of*** G. A graph whose vertices-set is empty is a ***null graph*** and a graph whose edge-set is empty is an ***empty graph***. Color- ing of a graph G is an assignment of colors (elements of some set) to the vertices of G, one color to each vertex, so that adjacent vertices are assigned distinct colors. If n colors are used, then the coloring is referred to as an $n-coloring$. If there exists an n-coloring of a graph G, then G is called $n-colorable$. The minimum n for which a graph G is n-colorable is called the ***chromatic number*** of G, and is denoted by $\chi(G)$. The ***core*** of a graph G is the union of cycles in G. A vertex x of a graph G is called an ***end vertex*** if $deg(x) = 1$. For further study of graph theory see [(Bondy and Murty, 2008)].

1.2 Definition:

An element n of an le-module M is said to be a ***submodule element*** if $n + n, rn \leqslant n$ for all $r \in R$. We denote the set of all submodule elements of M by $Sub(M)$.

Submodule elements are the ones on which the theory of an le-module is being studied. Observe that if $n, m \in Sub(M)$ then $n + m \in Sub(M)$ and $rn \in Sub(M)$, and note that $n + n = n$ for all $n \in Sub(M)$.

1.3 Proposition:

Let M be an le-module. Then

(1) $n \geqslant 0$, for $n \in Sub(M)$.

(2) $n + m \geqslant n \vee m$ for $n, m \in Sub(M)$.

(3) If $n + m \neq n \vee m$ implies $n \quad m$ and $m \quad n$ for $n, m \in Sub(M)$.

(4) If $n \leqslant m$ then $k + n \leqslant k + m$ for all $n, k, m \in M$.

(5) If $n, m \in Sub(M)$ then $n \wedge m \in Sub(M)$.

Proof. (1) We have $rn \leqslant n$ for all $r \in R$. Hence $0n = 0 \leqslant n$ for $n \in Sub(M)$.

(2) $n \vee (n + m) = (n + 0) \vee (n + m) = n + (0 \vee m) = n + m$ and therefore $n + m \geqslant n$. Similarly $n + m \geqslant m$ and consequently, $n + m \geqslant n \vee m$.

(3) If $n + m \neq n \vee m$ then $n + m > n \vee m$. On contrary suppose that $n \leqslant m$ then $n \vee m = m$. Therefore we have $m < n+m$. But then $m + \vee\{m, n\} = m+m = m \neq n+m = \vee\{n+m, n+n\}$ implies that M is not an le-module. Hence $n \quad m$ and similarly $m \quad n$.

(4) By the definition of le-module, we have $(k + n) \vee (k + m) = k + (n \vee m) = k + m$.

(5) We have $n \wedge m \leqslant n$ then by (4), $n \wedge m + n \wedge m \leqslant n \wedge m + n \leqslant n + n = n$. Similary $n \wedge m + n \wedge m \leqslant m$. Consequently, $n \wedge m + n \wedge m \leqslant n \wedge m$. If $r \in R$ then $r(n \wedge m) \leqslant rn \leqslant n$ and also $r(n \wedge m) \leqslant m$ implies that $r(n \wedge m) \leqslant n \wedge m$. Hence $n \wedge m$ is submodule element.

□

In particular, if 0 is the smallest element in M, i.e., $0 \leqslant n$ for all $n \in M$ then $n + m \geqslant n \vee m$ for all $n, m \in M$

We call $p \in Sub(M)$ as ***prime element*** if for $r \in R$ and $n \in M, rn \leqslant p$ implies $re \leqslant p$ or $n \leqslant p$. We call $q \in Sub(M)$ as ***maximal element*** if $q \neq e$ and for any $n \in Sub(M)$ with $q \leqslant n \leqslant e$ either $q = n$ or $n = e$. We denote the set of all maximal submodule elements by $Max(M)$.

1.4 Lemma:

[(Bhuniya and Kumbhakar, 2018, 2019)] Let M be an le-module then every maximal element in M is prime.

1.5 Lemma:

Let $n, m \in Sub(M)$. If $n + m = e$ for all m n then $n \in Max(M)$.

Proof. If $n \notin Max(M)$ then there exists $k \in Sub(M)$ such that $k > n$. Therefore $n + k = e$ but as $k > n$ by Proposition [1.3(3)] we have $n + k = k$, a contradiction. □

In this paper we associate a graph $G(M)$ on $Sub(M)$ as follows: The vertex set of the graph $G(M)$ is the set of all nonzero proper submodule elements of M and two distinct vertices n and m are adjacent if $n + m = e$ and we call this graph as ***associated graph of le-module***.

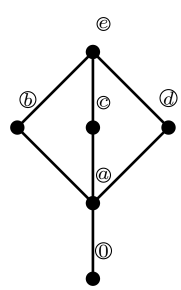
1.6 Remark:

Note that for $n, m \in G(M)$, if $n + m = e$ and $n \leqslant k$ then by Proposition [1.3(4)] $k + m = e$ hence $deg(n) \leqslant deg(k)$.

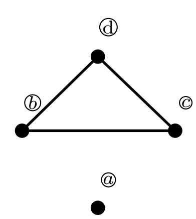
1.7 Example:

Let $M = \{0, a, b, c, d, e\}$ be an le-module over ring Z_2 with its lattice diagram and its graph with + is given in the table and usual multiplication with elements of the ring i.e $0x = 0$ and $1x = x$ for all $x \in M$. Note that $G(M) = \{a, b, c, d\}$ and $Sub(M) = M$.

+	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	a	b	c	d	e
b	b	b	b	e	e	e
c	c	c	e	c	e	e
d	d	d	e	e	d	e
e	e	e	e	e	e	e



M



$G(M)$

1.8 Definition:

A submodule element n of an le-module M is called ***super uous (or small)*** if for every $m \in Sub(M)$, $n + m = e$ implies $m = e$. Note that 0 is always superfluous element but e is not superfluous.

1.9 Definition:

An le-module M is said to be a multiplication le-module if every submodule element n of M can be expressed as $n = Ie$ for some ideal I of R, where $Ie = \vee\{\sum_{i=0}^{k} r_i e | k \in \mathsf{N}, r_i \in I\}$ [(Bhuniya and Kumbhakar, 2018, 2019)]

1.10 Lemma:

M is multiplication le-module if and only if $n = (n : e)e, \forall n \in Sub(M)$ where $(n : e) = \{r \in R | re \leqslant n\}$.[(Bhuniya and Kumbhakar, 2018, 2019)]

1.11 Definition:

An le-module M is called ***coatomic*** if for every n of M there exists $m \in Max(M)$ such that $n \leqslant m$.

1.12 Definition:

An le-module M is called ***simple*** if 0 and e are the only submodule elements of M.

1.13 Definition:

The radical of an le-module M is the smallest submodule element if exists say, $n \in M$, such that $n \geqslant m$ for every superfluous element m of M. Otherwise it is equal to e. We denote the radical of an le-module M by $Rad(M)$.

2. MAIN RESULTS:

2.1 Lemma:

Let M be a coatomic le-module then $Rad(M)$ is a superfluous element and $Rad(M) = \wedge_{m \in Max(M)} m$.

Proof. Let l be a superfluous element and there exists $m \in Max(M)$ with l m. Then $l + m = e$ and l being superfluous implies $m = e$, a contradiction. Therefore $l \leqslant m$ for all $m \in Max(M)$ and hence $l \leqslant \wedge_{m \in Max(M)} m$ for any superfluous element l. Thus we have $Rad(M) \leqslant \wedge_{m \in Max(M)} m$. It is enough to prove that $\wedge_{m \in Max(M)} m$ is a superfluous element. Suppose $\wedge_{m \in Max(M)} m + n = e$ for some $n \in Sub(M)$ then $n \leqslant m_j$ for some $m_j \in Max(M)$ then $\wedge_{m \in Max(M)} m + m_j = e$ but $\wedge_{m \in Max(M)} m \leqslant m_j$ implies $\wedge_{m \in Max(M)} m + m_j = m_j$, a contradiction. Hence $\wedge_{m \in Max(M)} m$ is a superfluous element and hence $Rad(M) = \wedge_{m \in Max(M)} m$ □

2.2 Theorem:

Let M be a coatomic le-module and $n, m \in Sub(M)$. Then $n \leqslant Rad(M)$ if and only if n is superfluous.

Proof. Suppose that $n \leqslant Rad(M)$ and n is not superfluous i.e., there exists $m \in Sub(M)$ with $n + m = e$ and $m < e$. Since M is coatomic, there exists $k \in Max(M)$ with $m \leqslant k$. This implies that $n + m \leqslant k + k = k < e$, a contradiction. Consequently $m = e$.

Conversely suppose that n is superfluous and n $Rad(M)$. Therefore there exists $m \in Max(M)$ with n m. Which implies $n + m = e$ and since n is superfluous we have $m = e$, a contradiction. Consequently $n \leqslant Rad(M)$. □

2.3 Proposition:

[(Bhuniya and Kumbhakar, 2018, 2019)] Let M be an le-module and $x \in M$. Then for submodule elements k, l, n of M,

(1) $l \leqslant n$ implies $(l : x) \subseteq (n : x)$ and $(k : n) \subseteq (k : l)$;
(2) $(l \wedge n : k) = (l : k) \cap (n : k)$

2.4 Theorem:

Let M be a multiplication le-module and $n, m \in Sub(M)$. If p is a prime submodule element of M with $n \wedge m \leqslant p$, then $n \leqslant p$ or $m \leqslant p$.

Proof. Suppose that p is a prime submodule element of M with $n \wedge m \leqslant p$. Then by Proposition[2.3] $(n \wedge m : e) \subseteq (p : e)$ therefore $(n \wedge m : e) = (n : e) \cap (m : e) \subseteq (p : e)$. Let $r_1 \in (n : e)$ and $r_2 \in (m : e)$ with $r_1, r_2 \notin (p : e)$. Then $r_1 e \leqslant n$ and $r_2 e \leqslant m$ and this implies

$r_1r_2e \leqslant n \wedge m \leqslant p$. But since p is prime, we have $r_1e \leqslant p$ or $r_2e \leqslant p$, i.e., $r_1 \in (p:e)$ or $r_2 \in (p:e)$, a contradiction. Therefore $(n:e) \subseteq (p:e)$ or $(m:e) \subseteq (p:e)$. Now, since M is a multiplication le-module, by Lemma[1.11] we have $n = (n:e)e \leqslant (p:e)e = p$ or $m = (m:e)e \leqslant (p:e)e = p$. Consequently $n \leqslant p$ or $m \leqslant p$. □

2.5 Corollary:

Let M be a coatomic multiplication le-module with $Max(M) = \{m_i | i \in I\}$ and 0 as smallest element. Then for any nonempty proper finite subset $\wedge$ of I, there exists proper submodule element, say m such that $\wedge_{i\in I} m_i + m = e$

Proof. On contrary suppose that there does not exists such m. Then by Theorem[2.2] $\wedge_{i\in\wedge} m_i \leqslant Rad(M) \leqslant m_j$ for all $j \in I$ and $j \notin \wedge$. Since every maximal element is a prime element by Lemma[1.5], then by Theorem[2.4] we have $m_i \leqslant m_j$ for some $i \in \wedge$, a contradiction. □

Note that if $\{M_i | 1 \leqslant i \leqslant n\}$ is a family of le-modules over ring R then $M = \prod_{i=1}^n M_i$ is also an le-module over R with coordinate-wise addition, scalar multiplication and ordering. Also if $m_i \in Sub(M_i)$ for $i = 1, 2, \ldots n$ then $\prod_{i=1}^n m_i \in Sub(M)$. In the following theorem we have discussed the graph structure on product of simple le-modules

2.6 Theorem:

Let $M = \prod_{i=1}^n M_i$ where each M_i is a simple le-module then $G(M)$ is a connected $n-$partite graph.

Proof. Let 0 and e_i be the only submodule elements of M_i for $i = 1, 2, \ldots, n$. Then note that $Sub(M) = \{\prod_{i=1}^n a_i | a_i = 0 \text{ or } e_i\}$. Let $\prod_{i=1}^n a_i$ and $\prod_{i=1}^n b_i$ be any two nonzero proper submodule elements of M. If $\prod_{i=1}^n a_i + \prod_{i=1}^n b_i = \prod_{i=1}^n e_i$ then $\prod_{i=1}^n a_i$ and $\prod_{i=1}^n b_i$ are adjacent. Suppose that they are not adjacent. Then we have following two cases:

Case i) If there exists j such that $a_j = b_j = e_j$ then $\prod_{i=1}^n c_i$ is a proper submodule element of M with $c_i = e_i$ for $i \neq j$ and $c_j = 0$ for $i = j$. Then $\prod_{i=1}^n a_i$ and $\prod_{i=1}^n b_i$ are both adjacent to $\prod_{i=1}^n c_i$

Case ii) If there does not exist j such that $a_j = b_j = e_j$ then take $\prod_{i=1}^n c_i$ such that $c_i = 0$ if $a_i = e_i$ and $c_i = e_i$ if $a_i = 0$. Similarly, choose submodule element $\prod_{i=1}^n d_i$ related to $\prod_{i=1}^n b_i$. Then $\prod_{i=1}^n a_i$ is adjacent to $\prod_{i=1}^n c_i$ and $\prod_{i=1}^n b_i$ is adjacent to $\prod_{i=1}^n d_i$. Note that $\prod_{i=1}^n c_i$ and $\prod_{i=1}^n d_i$ are adjacent because $c_i = d_i = 0$ and this will imply $a_i = b_i = e_i$.

Therefore $G(M)$ is a connected graph. Now let $V_j = \prod_{i=1}^n m_i$ with $m_i = e_i$ for $0 \leqslant i \leqslant j-1$, $m_j = 0$ and $m_i = 0$ or e_i for $j+1 \leqslant i \leqslant n$. Hence no two vertices of V_i are adjacent. Consequently $G(M)$ is a connected $n-$partite graph. □

Sachin Ballal and Vilas Kharat studied Zariski topology on lattice modules [(Ballal and Kharat, 2015)]. In [(Ballal and Kharat, 2019)], they have topologize minimal spectrum of multiplication lattice modules . In [(Bhuniya and Kumbhakar, 2018)], Bhuniya and Kumbhakar studied prime spectrum of an le-module. Let $Spec(M) = \{p \in M | p \text{ is a prime submodule element of } M\}$. For $n \in Sub(M)$, we denote $V(n) = \{p \in Spec(M) : n \leqslant p\}$ and $\nu(M) = \{V(n) | n \in Sub(M)\}$. If $\nu(M)$ is closed under finite unions then there exists a topology on $Spec(M)$ and we call this topology the quasi-Zariski topology and in this case M is called the top le-module. Note that associated graph on $Spec(M)$ is a subgraph of $G(M)$ and we denote it by $G^{Spec}(M)$.

In the following result, we establish a relationship between topology on $Spec(M)$ and the graph on it.

2.7 Theorem:

Let M be a non-primeless top le-module. Then $G^{Spec}(M)$ is a complete graph if and only if $Spec(M)$ is a T_1-space.

Proof. Suppose $G^{Spec}(M)$ is a complete graph. Let $p, q \in Spec(M)$ with $q \in V(p)$. This implies $p \leqslant q$. If $p \neq q$ and $G^{Spec}(M)$ is a complete graph then $p + q = e$ implies $q = e$. For $p \leqslant q$ implies $q \leqslant p + q \leqslant q + q = q$, i.e. $p + q = q$. But $q \neq e$ implies $V(p) = \{p\}$ is a closed set. Consequently $Spec(M)$ is a T_1-space.

Conversely, suppose that $Spec(M)$ is a T_1-space. Therefore, $\{p\}$ is closed for all $p \in Spec(M)$. Note that $\{p\} = \cap_j V(q_j)$, where $j \in I$ for some index set I and if $p \leqslant q$ then $q_j \leqslant p \leqslant q$ for all j. This implies $q \in \cap_j V(q_j)$ and therefore $p = q$ and hence $V(p) = \{p\}$. Thus every prime submodule element is maximal and hence $p \neq q$ implies $p + q = e$. Consequently $G^{Spec}(M)$ is a complete graph. □

Next Corollary establish the relation between $Spec(R)$ and $G^{Spec}(M)$.

2.8 Corollary:

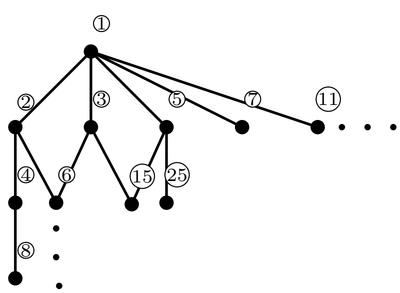
Let M be non-primeless multiplication le-module. If $Spec(R)$ is a T_1-space then $G^{Spec}(M)$ is a complete graph.

Proof. Suppose that $p, q \in Spec(M)$ with $p \in \overline{\{q\}}$. Then $V(q) \supseteq \overline{\{q\}}$ which implies $q \leqslant p$ and therefore $(q : e) \subseteq (p : e)$. Since $Spec(R)$ is a T_1-space we have $(q : e) = (p : e)$ and therefore by Lemma[1.11] $q = (q : e)e = (p : e)e = p$ since M is a multiplication le-module. Hence $\overline{\{q\}} = \{q\}$ i.e., $\{q\}$ is a closed set in $Spec(M)$. Therefore $Spec(M)$ is a T_1-space. By Theorem[2.7] $G^{Spec}(M)$ is a complete graph. □

The Converse of the Corollary[2.8] is not necessarily true. See the following example.

2.9 Example:

Let $M = \mathsf{Z}$ over $R = \mathsf{Z}$ be a multiplication le-module with respect to $n + m = g.c.d(n, m)$, rn is the usual multiplication for $r \in R$, $n \in M$ and $m \geqslant n$ if and only if m divides n. Here note that $Spec(R) = \{p\mathsf{Z} | p \text{ is a prime number }\}$ and $Spec(M) = \{p \in \mathsf{Z} | p \text{ is a prime number }\}$. Note that $G^{Spec}(M)$ is a complete graph even though $Spec(R)$ is not T_1-space.



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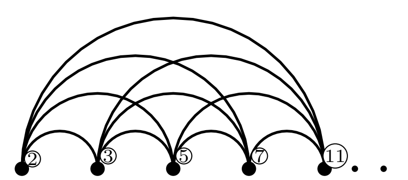


$G^{Spec}(M)$

2.10 Theorem:

Let M be an le-module. Then $Rad(M) = 0$ if and only if $G(M)$ is connected.

Proof. Suppose that $Rad(M) = 0$ and $m_1, m_2 \in G(M)$ with $m_1 \neq m_2$. Note that m_1 and m_2 are not superfluous. Then there exist nonzero proper submodule elements n_1 and n_2 with $m_1 + n_1 = e$ and $m_2 + n_2 = e$. Then m_1 and n_1 are adjacent and also m_2 and n_2 are adjacent. If $n_1 + n_2 = e$ then n_1 and n_2 are adjacent and there is a path between m_1 and m_2. If $n_1 + n_2 \neq e$ then $m_1 + (n_1 + n_2) = e$ and $m_2 + (n_1 + n_2) = e$ and therefore there is a path between m_1 and m_2. Consequently, $G(M)$ is connected.

Conversely, suppose $G(M)$ is connected and $Rad(M) \neq 0$. Then there exists nonzero superfluous element m such that $m \leqslant Rad(M)$. Since m is superfluous $m + l = e$ for some $l \in Sub(M)$ implies $l = e$ i.e., m is an isolated vertex of $G(M)$, a contradiction. Therefore $Rad(M) = 0$ □

2.11 Corollary:

Let M be a coatomic le-module. If $G(M)$ is a tree then $|Max(M)| = 2$.

Proof. Since $G(M)$ is a tree, $G(M)$ has no cycle which implies $|Max(M)| < 3$ and by Theorem[2.10] $G(M)$ is connected implies $Rad(M) = 0$ and therefore $|Max(M)| > 1$, consequently $|Max(M)| = 2$ □

Following Theorem establish a Beck's conjecture for coatomic le-module.

2.12 Theorem:

Let M be a coatomic le-module. Then the clique number and the chromatic number of $G(M)$ are equal to $|Max(M)|$.

Proof. Let S be a complete subgraph of $G(M)$. For each vertex n of S there exists a maximal element m_n with $n \leqslant m_n$. For distinct vertices n_1 and n_2 of S, since $n_1 + n_2 = e$, we have $m_{n_1} + m_{n_2} = e$ and which implies $m_{n_1} \neq m_{n_2}$. Thus the subgraph induced by $T = \{m_n | n \text{ is a vertex of } S\}$ is a complete graph and $|S| \leqslant |T|$. Now $G^{Max}(M)$ is a complete subgraph and $|S| \leqslant |G^{Max}(M)|$ for complete subgraph S. Hence the clique number of $G(M) = |Max(M)| = |G^{Max}(M)|$

Now to find the chromatic number of $G(M)$, let $\{m_\lambda | \lambda \in \wedge\}$ be the set of all maximal submodule elements of M. For any $\lambda \in \wedge$, let $G_\lambda(M) = \{n \in Sub(M) | 0 \neq n \leqslant m_\lambda, n \notin \bigcup_{\lambda' < \lambda} G_{\lambda'}(M)\}$. Then for $\lambda \in \wedge$, $m_\lambda \in G_\lambda(M)$ and $G_\lambda(M) \neq$?. Also $\{G_\lambda(M) | \lambda \in \wedge\}$ forms a partition for the set of all vertices of $G(M)$. Since for every $\lambda \in \wedge$, any two vertices in $G_\lambda(M)$ are not adjacent, all vertices in $G_\lambda(M)$ can have the same colour. However the $m_\lambda' s$ must have different colours. Consequently the chromatic number of $G(M) = |\wedge|$. □

2.13 Definition:

Two vertices n and m are said to be orthogonal in $G(M)$ if $n + m = e$ and for every $k \in G(M)$ either $n + k \neq e$ or $m + k \neq e$.

2.14 Theorem:

If M is a coatomic le-module then the following statements are equivalent;

(1) $G(M)$ has no triangle.

(2) Every two adjacent submodule elements are orthogonal.

(3) M has at most two maximal submodule elements.

Proof. (1) $\implies$ (2) Suppose $n, m \in G(M)$ with $n + m = e$. If $m + k = e, n + k = e$ for some $k \in G(M)$ then $n - m - k - n$ forms a triangle in $G(M)$ and therefore either $n + k \neq e$ or $m + k \neq e$.

(2) $\implies$ (3) If $|Max(M)| \geqslant 3$. Let $m_1, m_2, m_3 \in Max(M)$ be distint elements then it forms triangle of maximal submodule elements $m_1 - m_2 - m_3 - m_1$ and we get non-orthogonal adjacent vertices $\{m_1, m_2\}$.

(3) $\implies$ (1) If $|Max(M)| = 1$ then we get empty graph. If $|Max(M)| = 2 = |G(M)|$ then our graph is K_2. Now suppose $|G(M)| \geqslant 3$ and $Max(M) = \{m_1, m_2\}$ then for $n \in Sub(M)$ we have $n \leqslant m_1$ or $n \leqslant m_2$. Thus any two submodule elements of M are $\leqslant m_1$ or $\leqslant m_2$. Without loss of generality if $n \leqslant m_1$ and $k \leqslant m_1$ then $n + k \leqslant m_1 \neq e$. Therefore $G(M)$ has no triangle. □

2.15 Corollary:

In a coatomic le-module the girth of $G(M)$ is always 3 except when $|Max(M)| \leqslant 2$.

2.16 Definition:

Let M be an le-module such that $Max(M) \neq$? and let $S = \{n \in G(M) | n \quad Rad(M)\}$. The subgraph generated by the set S is deonoted by $G^*(M)$.

Note that, if $Rad(M) = 0$ then $G^*(M) = G(M)$

2.17 Theorem:

Let M be an le-module with $Max(M) \neq$? . Then $G^*(M)$ is connected and $diam(G^*(M)) \leqslant 3$.

Proof. Let m_1 and m_2 be two distinct elements of $G^*(M)$. Then there exist $n_1, n_2 \in Max(M)$ such that $m_1 \quad n_1, m_2 \quad n_2$. If $n_1 = n_2$ then $m_1 + n_1 = e = m_2 + n_1$ and we have a path between m_1 and m_2. If $n_1 \neq n_2$ then $n_1 + n_2 = e$ and we again have a path between m_1 and m_2. □

2.18 Theorem:

Let M be an le-module with $|Max(M)| = \infty$. Then there exists $n \in G^*(M)$ such that $|Max(M) \backslash \nu(n)| = \infty$, where $\nu(n) = \{m \in Max(M) | n \leqslant m\}$.

Proof. If possible, suppose that for every $n \in G^*(M)$, $|Max(M) \backslash \nu(n)| < \infty$. Let m_1 and m_2 be two distinct elements of $G^*(M)$. Then $|Max(M) \backslash \nu(m_1)| < \infty$ and $|Max(M) \backslash \nu(m_2)| < \infty$ and implies $|\nu(m_1) \cap \nu(m_2)| = \infty$. Therefore there exists $q \in Sub(M)$ with $m_1 \leqslant q$ and $m_2 \leqslant q$. But then $m_1 + m_2 \leqslant q + q = q \neq e$ and which implies $G^*(M)$ is totally disconnected, a contradiction to Theorem[2.17]. □

2.19 Theorem:

Let M be a coatomic le-module. Then the following statements are equivalent.

(1) $G^*(M)$ is a complete bipartite graph.
(2) $|Max(M)| = 2$.

Proof. Suppose that $G^*(M)$ is a complete bipartite graph with two parts say V_1 and V_2. If $|Max(M)| = 1$ then $G^*(M)$ cannot be bipartite. Therefore $|Max(M)| \geqslant 2$. Suppose that $|Max(M)| > 2$ then by the pigeon-hole principle two of the maximal elements belong to one of V_i, a contradiction to $G^*(M)$ is a complete bipartite graph.

Conversely, suppose that $Max(M) = \{n_1, n_2\}$. Since M is coatomic, every submodule element of M is $\leqslant n_1$ or $\leqslant n_2$. Let $V_1 = \{n \in G^*(M) | n \leqslant n_1\}$ and $V_2 = \{n \in G^*(M) | n \leqslant n_2\}$. If $n \in V_1 \cap V_2$ then $n \leqslant n_1 \wedge n_2 = Rad(M)$. But since $n \in G^*(M)$, $n \quad Rad(M)$. Therefore $V_1 \cap V_2 =$? and $G^*(M) = V_1 \cup V_2$. Let $m_1 \in V_1$ and $m_2 \in V_2$ then $m_1 + m_2 \quad n_1$, otherwise $m_2 \leqslant n_1$, which is not true. Similarly $m_1 + m_2 \quad n_2$. Consequently, $m_1 + m_2 = e$ and $G^*(M)$ is a complete bipartite graph. □

2.20 Corollary:

Let M be a coatomic le-module with $|Max(M)| > 1$ then $G^*(M)$ is a star graph or girth of $G(M) \leqslant 4$.

Proof. Let M be a coatomic le-module with $|Max(M)| = 2$ then by Theorem[2.20], $G^*(M)$ is a complete bipartite graph. Let V_1 and V_2 be two parts of the graph $G^*(M)$. If V_1 or V_2 contains single element then $G^*(M)$ is a star graph. Otherwise we have a cycle $v_{11} - v_{21} - v_{12} - v_{22} - v_{11}$ for $v_{11}, v_{12} \in V_1$ and $v_{21}, v_{22} \in V_2$ and hence the girth of $G(M)$ is 4. If $|Max(M)| > 2$ then by Corollary[2.15], the girth of $G(M)$ is 3. □

2.21 Theorem:

If M is a coatomic le-module and $|Max(M)| = n < \infty, n > 1$ then $G^*(M)$ is $n-$partite

Proof. Suppose that $Max(M) = m_1, m_2, \ldots, m_n$. Since M is coatomic, every submodule element is $\leqslant m_i$ for some $i \in \{1, 2, \ldots, n\}$. Let $W_i = \{n \in G^*(M) | n \leqslant m_i\}$ then take $V_i =$

$W_i \setminus \cup_{j=1}^{i-1} W_j$. If $m_{i1}, m_{i2} \in V_i$ and $m_{i1} + m_{i2} = e$ then $e = m_{i1} + m_{i2} \leqslant m_i + m_i = m_i$, a contradiction. Also $m_i \in V_i$ implies $V_i \neq$? and $M = V_1 \cup V_2 \cup \ldots \cup V_n$. Consequently, $G^*(M)$ is $n-$partite □

2.22 Theorem:

Let M be an le-module and $G^*(M)$ be a star graph. Then $|Max(M)| = 2$ and M is coatomic.

Proof. Since $G^*(M)$ is a star graph implies $|Max(M)| < 3$ and there exists a vertex $n \in G^*(M)$ such that n is adjacent to all other vertices. Then $n \in Max(M)$. For, if $k \in G(M)$ with $n \leqslant k$ then $n + k \leqslant k + k = k \neq e$ and it implies that n and k are not adjacent and therefore $k \notin G^*(M)$, a contradiction to the fact that $n \in G^*(M)$. Note that $|Max(M)| \neq 1$, otherwise $Rad(M) = n$ and $n \notin G^*(M)$. Now suppose that $m \in Max(M)$. If s n and s m for some $s \in Sub(M)$ then $s + n = e, s + m = e$ then $s \in G^*(M)$ and $s - n - m - s$ is a cycle in $G^*(M)$, a contradiction. Hence M must be coatomic and $|Max(M)| = 2$. □

2.23 Theorem:

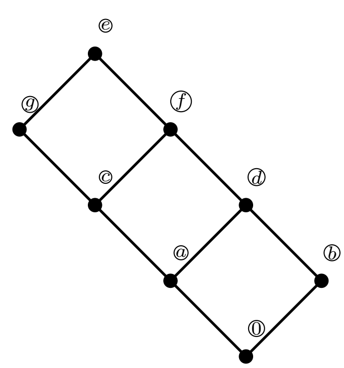
If M is coatomic le-module then $G^*(M)$ is a star graph if and only if $G^*(M)$ is a tree graph.

Proof. If $G^*(M)$ is a star graph then clearly it is a tree graph. Conversely suppose that $G^*(M)$ is a tree graph then $|Max(M)| < 3$ because tree contains no cycle. If $|Max(M)| = 1$ then the graph of $G^*(M)$ is empty. If $|Max(M)| = 2$ then by Theorem[2.19], $G^*(M)$ is a complete bipartite graph. Therefore $G^*(M)$ is complete bipartite and tree. Hence $G^*(M)$ is a star graph □

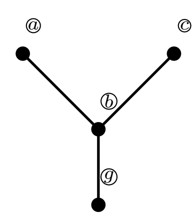
2.24 Example:

Let $M = \{0, a, b, c, d, g, f, e\}$ be an le-module over Z_2 with '+' as given in the table and usual multiplication of ring Z_2 with elements of M ,i.e., $0x = 0$ and $1x = x$ for all $x \in M$ and note that $Sub(M) = \{0, a, b, c, g, e\}$. Here $G^*(M) = G(M)$ as $Rad(M) = 0$ and $G^*(M)$ is a star graph and hence complete bipartite graph as shown in the following Figure.

+	0	a	b	c	d	g	f	e
0	0	a	b	c	d	g	f	e
a	a	a	e	c	e	g	e	e
b	b	e	b	e	e	e	e	e
c	c	c	e	c	e	g	e	e
d	d	e	e	e	e	e	e	e
g	g	g	e	g	e	g	e	e
f	f	e	e	e	e	e	e	e
e	e	e	e	e	e	e	e	e



M



$G^*(M) = G(M)$

2.25 Theorem:

If M is a coatomic le-module and $G^*(M)$ contains a cycle then the core of G is a union of triangles and rectangles, and every vertex of $G^*(M)$ is either an end vertex or a vertex of the core.

Proof. Suppose $(m_1, m_2, \ldots, m_n, m_1)$ is a cycle. If $n \leqslant 4$ then the result holds trivially. If $n \geqslant 5$ and $m_1 + m_3 = e$ or $m_2 + m_{n-1} = e$ or $m_2 + m_n = e$ then $m_1 - m_2$ belongs to a triangle or rectangle. Assume that $m_1 + m_3 \neq e$, $m_2 + m_{n-1} \neq e$ and $m_2 + m_n \neq e$.

Case i) Suppose $m_1 + m_{n-1} \neq e$. Then there exists $m \in Max(M)$ with $m_1 + m_{n-1} \leqslant m$ and hence $m_1 + m_{n-1} + m_2 \leqslant m + m_2$ and therefore $m + m_2 = e$. Similary, $m + m_n = e$ and consequently (m_1, m_2, m, m_n, m_1) is a rectangle.

Case ii) Suppose $m_1 + m_{n-1} = e$. Here we will use mathematical induction. If $n = 5$ then $(m_1, m_2, m_3, m_4, m_1)$ is a cycle. Assume the result for $n = k$. Then, $(m_1, m_2, \ldots, m_{k-1}, m_1)$ we have $m_1 - m_2$ belongs to a triangle or a rectangle. If $(m_1, m_2, \ldots, m_{k-1}, m_k, m_1)$ is a cycle then by assumption $m_1 + m_{k-1} = e$ and which implies that $(m_1, m_2, \ldots, m_{k-1}, m_1)$ is a cycle and therefore by the induction result follows.

Suppose that m is not a vertex of a cycle. We prove that only one edge is adjacent to m. If possible, suppose m_1, m_2 are two vertices adjacent to m then there exists a path $m_1 - m - m_2 - c$ since $G^*(M)$ is connected. If $m_3 = m_1 + c$ and $m_3 = e$, then (m_1, m, m_2, c, m_1) is a cycle and therefore $m_3 \neq e$. But then $m_2 + m_3 = m_2 + m_1 + c = e + m_1 = e$ and this implies $m + m_3 = m + m_1 + c = e + c = e$, consequently (m, m_2, m_3, m) is a cycle, a contradiction. Therefore m is an end vertex. $\square$

2.26 Theorem:

[(Bhuniya and Kumbhakar, 2019)] Let p be a prime submodule element of an le-module M and $x \in M$. Then $(p : x)$ is a prime ideal of R for every $x \in M$.

2.27 Theorem:

Let M be a coatomic le-module over a ring R. If for $n, m \in Max(M)$ with $(m : e)$ * $(n : e)$ then $G^*(M)$ is complete if and only if $G^*(M) = K_2$.

Proof. If $|Max(M)| = 1$ then $G^*(M)$ is an empty graph. Suppose $|Max(M)| > 2$. Let $m_1, m_2, m_3 \in Max(M)$ be distinct elements. We prove that $(m_1 : e)m_2 \in G^*(M)$. If possible, suppose $(m_1 : e)m_2 \leqslant Rad(M)$ then $(m_1 : e)m_2 \leqslant m_3$. Let $r_1 \in (m_1 : e)$ and $r_2 \in (m_2 : e)$ implies $r_1 e \leqslant m_1$ and $r_2 e \leqslant m_2$. Therefore $r_1 r_2 e \leqslant r_1 m_2 \leqslant m_3$ implies $r_1 r_2 \in (m_3 : e)$ and hence $(m_1 : e)(m_2 : e) \subseteq (m_3 : e)$. But m_3 is a maximal element and hence prime, by Theorem[2.26], $(m_3 : e)$ is a prime ideal. Therefore $(m_1 : e) \subseteq (m_3 : e)$ or $(m_2 : e) \subseteq (m_3 : e)$, a contradiction to the assumption. Hence $(m_1 : e)m_2$ m_3 and therefore we have $(m_1 : e)m_2 \in G^*(M)$. Note that $(m_1 : e)m_2 \leqslant m_1$ implies $(m_1 : e)m_2 + m_1 = m_1 \neq e$. Hence $d((m_1 : e)m_2, m_1) \neq 1$. Now $(m_1 : e)m_2 + m_3 = e$ and therefore $d((m_1 : e)m_2, m_1) = 2$, but $diam(G^*(M)) = 1$ because $G^*(M)$ is complete. Therefore $|Max(M)| = 2$. Then by Theorem[2.19], $G^*(M)$ is a complete bipartite graph. Thus $G^*(M) = K_2$, because it is complete and complete bipartite graph.

$\square$

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