Characterizations of deletable elements and reducibility numbers in Some Classes of lattices

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In this paper, we have obtained some characterizations of deletable elements and studied reducibility in chains, graded, complete, planar, algebraic, relatively atomic and locally modular lattices.

The notion of reducibility number introduced by Kharat et al. is also studied in these classes of lattices.

Keywords: Deletable elements, Reducibility, Reducibility numbers, Graded, Complete, Planar, Algebraic, Relatively Atomic, Locally Modular.

1. INTRODUCTION

The concept of reducibility introduced by Bordalo and Monjardet (Bordalo and Monjardet, 1996) is well studied for some classes of lattices. They have proved that the classes of lattices such as pseudocomplemented, upper semimodular, lower semimodular, upper locally distributive, lower locally distributive and (meet) semidistributive are reducible. It is easy to see that the lattice L depicted in Figure $1(a)$ has no deletable element with respect to the class of distributive lattices and also with respect to the class of modular lattices. Therefore, the class of distributive lattices as well as the class of modular lattices is not reducible.

Definition 1.1. [2] *An element x of a lattice L satisfying a property p is deletable if $L - x$ is a lattice satisfying p. A class of lattices is reducible "if each lattice with at least two elements of this class admits at least one deletable element."*

This very definition fits in if we replace the term lattice by the term poset (see Bordalo and Monjardet (Bordalo and Monjardet, 1996), Kharat and Waphare (Kharat and Waphare, 2001)). Equivalently, one can easily note that a class of lattices is reducible if and only if one can go from any lattice in this class to trivial lattice by a sequence of lattices of the class obtained by deleting one element at each step.

All the lattices considered here for studying deletability and reducibility are of finite length, unless otherwise stated. Also, the study of deletable elements, say x, has been carried out in this article essentially for $x \neq 0, 1$. However in general we also note that 0 is deletable only if 0 is meet-irreducible and 1 is deletable only if 1 is join-irreducible.

Following definition of reducibility number is introduced by Kharat et al. (Kharat, Waphare, and Thakare, 2007).

Definition 1.2 (Kharat et al., 2007). *Let $\mathcal{P}$ be a class of posets and $P \in \mathcal{P}$. We say that a non-empty subset S of P is deletable if $P - S \in \mathcal{P}$. A positive number r is called reducibility number of P with respect to the class $\mathcal{P}$, denoted by $red(P, \mathcal{P})$, if there exists a deletable subset S of P with $|S| = r$ and no non-empty subset T of P with $|T| < r$ is deletable.*

Remark 1.3 (Kharat et al., 2007). *A class $\mathcal{P}$ is a reducible class of posets if and only if for any $P \in \mathcal{P}$, $red(P, \mathcal{P}) = 1$.*

It cab be observed from following three theorems that the reducibility number of a given lattice may or may not be same with respect to different classes to which it belongs. In what follows, the class of distributive, modular and boolean lattices is respectively denoted by $\mathcal{D}$, $\mathcal{M}$ and $\mathcal{B}$.

THEOREM 1.4 (KHARAT ET AL., 2007). *For $n \geqslant 2$, $red(\mathbf{2}^n, \mathcal{D}) = 2^{n-2}$*

THEOREM 1.5 (KHARAT ET AL., 2007). *For $n \geqslant 2$, $red(\mathbf{2}^n, \mathcal{M}) = 2^{n-2}$*

THEOREM 1.6 (KHARAT ET AL., 2007). *For $n \geqslant 1$, $red(\mathbf{2}^n, \mathcal{B}) = 2^{n-1}$*

Bordalo et al. (Bordalo and Monjardet, 1996) proved that the class of distributive and modular lattices are not reducible. Hence we have the following.

REMARK 1.7. *If L is a distributive lattice, then $red(\mathbf{L}, \mathcal{D}) > 1$.*
If L is a modular lattice, then $red(\mathbf{L}, \mathcal{M}) > 1$.

REMARK 1.8. *The deletable elements in chains, graded, complete, planar, algebraic, relatively atomic and locally modular lattices are respectively denoted by $\mathcal{CH}$-deletable, $\mathcal{G}$-deletable, $\mathcal{CO}$-deletable, $\mathcal{PL}$-deletable, $\mathcal{AL}$-deletable, $\mathcal{RAT}$-deletable and $\mathcal{LM}$-deletable.*

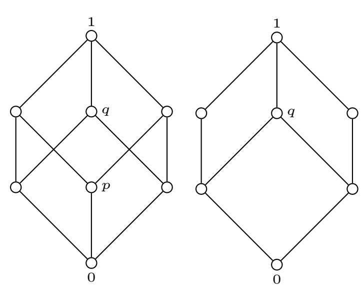
REMARK 1.9. *We consider the lattices with the least element 0 and the greatest element 1, if finite or bounded. However, we assume that 0 is deletable only if 0 is meet-irreducible and 1 is deletable only if 1 is join-irreducible (definitions are in the next section).*

REMARK 1.10. *The class of finite atomic lattices as well as the relatively atomic lattices is reducible since every element other then 0 and 1 is deletable in any lattice in these classes.*

DEFINITION 1.11. *(Grätzer, 1998) A partially ordered set is relatively atomic (or strongly atomic) if for all $a < b$ there is an element c such that $a \prec c \leqslant b$ or, equivalently, if every interval $[a, b]$ is atomic. Every relatively atomic partially ordered set with a least element is atomic.*

DEFINITION 1.12. *(Grätzer, 1998) A lattice L is called atomic if L has the least element 0 and the interval $[0, a]$ contains an atom for each $a > 0$.*

Following remark follows from Lemma 2.1. This notion, however, is different from the notion of dismantlability for lattices (Rival (Rival, 1974)). The lattice depicted in Figure $1(a)$ is an example which not dismantlable but reducible in the class of strong lattices.



L

(a)

$L - p$

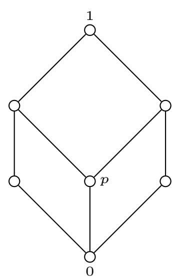
(b)

We note that, dismantlable lattices are the ones which can be completely dismantled by deleting one element at a time and resulting in a sublattice at each stage. As such, for a lattice L in the class of dismantlable lattices, one can reach to the trivial lattice by a sequence of sublattices of the class obtained by deleting one element at each step. Importantly, the lattices of the sequence of sublattices have no reason to preserve the properties of the original lattice. However, the class of dismantlable lattices is reducible.

In this connection, we recall a very important characterization of dismantlable lattices, known as Structure Theorem (see Thakare-Pawar-Waphare (Thakare, Pawar, and Waphare, 2002)).

THEOREM 1.13. *(Thakare et al., 2002) A finite lattice is dismantlable if and only if it is an adjunct of chains.*

Consider the lattice L depicted in Figure 1(a) which is an upper semimodular lattice. The resulting lattice depicted in Figure 1 (b) is also an upper semimodular lattice and therefore the element p is deletable . However, if we consider the lattice L depicted in Figure 1 (a) as a distributive lattice then the element p is not deletable as the resulting lattice $L-p$ depicted in Figure 1 (b) is not distributive. Also, in the pseudocomplemented lattice depicted in Figure 1 (a), the element q is not deletable. In fact, the element x has no pseudocomplement in $L-q$ (see Figure 2). (see to Grätzer (Grätzer, 1998), Birkhoff (Birkhoff, 1973), Stern (Stern, 1999), V. S. Kharat (Kharat, 2001))



$L-q$

2. CHARACTERIZATIONS OF DELETABLE ELEMENTS IN FINITE LATTICES

All the lattices in this section we assume are finite. The lattice theoretic definitions, concepts and relevant details can be found in (Birkhoff, 1973), (Grätzer, 1998), (Stern, 1991), (Stern, 1999),etc. An element $x \in L$ is *join-irreducible* if $x = a \vee b$ implies $x = a$ or $x = b$; it is *meet-irreducible* if $x = a \wedge b$ implies $x = a$ or $x = b$. An element which is both join and meet-irreducible is called doubly irreducible. The set $J(L)$ (respectively $M(L)$) shall denote the set of all non zero join-irreducible elements (non 1 meet-irreducible elements) of a given lattice L. Thus the set $J(L) \bigcap M(L)$ is the set of all doubly irreducible elements of L and the set $J(L) \bigcup M(L)$ is the set of all irreducible elements of L.

We also use the following notations for $x \in L$.
$x^{\curlyvee} = \{y \in L \mid x \prec y\}$ and
$x^{\curlywedge} = \{z \in L \mid z \prec x\}$.

It is immediate that if $x \in J(L)$ then $x^{\curlywedge}$ is a singleton set and so also $x^{\curlyvee}$ if $x \in M(L)$ and we shall denote these singleton sets simply by x^- and x^+ respectively. For $x \in L$, the depleted lattice $L - \{x\}$ is denoted by $L - x$ and for $a, b \in L - x$, if a is covered by b in $L - x$ in the induced binary operations on $L - x$, we shall denote the same by $a \prec^x b$.

Following result is a characterization of deletale elements in finite lattices.

LEMMA 2.1. *For an element $x \neq 0, 1$ of a finite lattice L, $L - x$ is a lattice if and only if $x \in L$ satisfies one of the following conditions.*

(1) $x \in J - M$
(2) $x \in M - J$
(3) $x \in J \bigcap M$.

Proof. Let L be a lattice and $x \in L$. Suppose that $L - x$ is a lattice and x does not satisfy any of the conditions $(i), (ii), (iii)$. Then there exist elements x_1, x_2, x_3, x_4 different from x in L with $x_1 \parallel x_2$ and $x_3 \parallel x_4$ such that $x_1 \vee x_2 = x = x_3 \wedge x_4$ in L. As $L - x$ is a lattice and $x_1 \parallel x_2$, suppose $w = g.l.b.\{x_3, x_4\}$ in $L - x$, and so in L $w \leqslant x_3, w \leqslant x_4$ and accordingly $w < x$. Note that, in $L - x$, $x_1 \leqslant x_3, x_4$ which implies that $x_1 \leqslant g.l.b.\{x_3, x_4\}$, i.e., $x_1 \leqslant w$. Similarly, $x_2 \leqslant x_3, x_4$ which implies that $x_2 \leqslant g.l.b.\{x_3, x_4\}$, i.e., $x_2 \leqslant w$ and so in L, $x_1 \leqslant w, x_2 \leqslant w$ which implies that $x_1 \vee x_2 \leqslant w$, i.e., $x \leqslant w$. However, $w < x$ and $x \leqslant w$ cannot both hold and consequently x must satisfy at least one of the conditions.

Conversely, suppose that $x \in L$ satisfies (i), i.e., $x \in J - M$:
Note that, as $x \in J$, join of every pair is preserved in $L - x$. Now, consider a pair of elements $x_1(\neq x), x_2(\neq x)$ such that $x_1 \wedge x_2 = x$ in L and such pair exists because $x \notin M$. In this case, $g.l.b.\{x_1, x_2\} = x^-$ in $L - x$ and meet of every pair other than such pairs will be preserved in $L - x$. Hence in this case $L - x$ is a lattice.
If x satisfies (ii), i.e., $x \in M - J$, then the result follows from dual arguments of case (i).
And, if x satisfies (iii), i.e., $x \in J \cap M$, this case follows from either case (i) or case (ii).
Hence in each case, $L - x$ forms a lattice. □

Definition 2.2 (Maeda, 1974). *A lattice L is called locally modular when there exists a congruence relation θ in L satisfying the following three conditions: (θ_1) If $a \neq 1$ in L then there exists $b \in L$ such that $b > a$ and $b \equiv a(\theta)$, and if $a \neq 0$ then there exists $b \in L$ such that $b < a$ and $b \equiv a(\theta)$. (θ_2) lf $a \prec b$ then $a \equiv b(\theta)$. (θ_M) For any $a \in L$, the sublattice $[a]$ is modular. L is called locally distributive (i.e. upper locally distributive) when, in the above definition, (θ_M) is replaced by the following condition: (θ_D) For any $a \in L$, the sublattice $[a]$ is distributive. Evidently, any locally distributive lattice is locally modular.*

Theorem 2.3 (Maeda, 1974). *Any locally modular lattice L is both upper and lower semimodular in the sense of Birkhoff.*

Definition of semimodular lattice in the above theorem and in the paper by Bordalo, et al. (Bordalo and Monjardet, 1996) is same.

Theorem 2.4 (Bordalo and Monjardet, 1996). *The class of upper locally distributive lattices is reducible. Moreover an element of a locally modular lattices is $\mathcal{ULD}$-deletable if and only if it is $\mathcal{USM}$-deletable.*

Hence we have the following Corollary.

Corollary 2.5. *The class of locally modular finite lattices is reducible. Moreover an element of a locally modular lattices is $\mathcal{LM}$-deletable if and only if it is $\mathcal{USM}$-deletable.*

As such, we have the following remark about the reducibility number of locally modular lattices.

Remark 2.6. *If $\mathbf{L}$ is a locally modular lattice, then $red(\mathbf{L}, \mathcal{L}M) = 1$.*

Bordalo et al. (Bordalo and Monjardet, 1996) also proved that the class of pseudocomplemented, lower (respectively, upper) semimodular and lower (respectively, upper) locally distributive lattices are reducible. Hence we have

Remark 2.7. *If L is a pseudocomplemented lattice, then $red(\mathbf{L}, \mathcal{P}C) = 1$.
If L is a lower semimodular lattice, then $red(\mathbf{L}, \mathcal{L}SM) = 1$.
If L is a upper semimodular lattice, then $red(\mathbf{L}, \mathcal{U}SM) = 1$.
If L is a lower locally distributive lattice, then $red(\mathbf{L}, \mathcal{L}LD) = 1$.
If L is a upper locally distributive lattice, then $red(\mathbf{L}, \mathcal{U}LD) = 1$.*

Definition 2.8. *(Grätzer, 1998) A lattice $(L, \leqslant)$ is called graded, sometimes ranked (but see Ranked poset for an alternative meaning), if it can be equipped with a rank function r from L to $\mathbb{N}$, sometimes to $\mathbb{Z}$, compatible with the ordering (so $r(x) < r(y)$ whenever $x < y$) such that* *whenever y covers x, then $r(y) = r(x) + 1$. The value of the rank function for an element is called its rank.*

Obviously, every chain is graded and so we have the following.

Remark 2.9. *Every element in a chain is deletable and so the class of chains is reducible.*

Remark 2.10. *If $\mathbf{L}$ is a chain, then $red(\mathbf{L}, \mathcal{C}H) = 1$.*

Following is a characterization of deletable elements in graded lattices that are not chains.

Theorem 2.11. *Let L be a (finite) graded lattice (other than chains) and $x(\neq 0,1) \in L$. Then x is $\mathcal{G}$-deletable if and only if $x \in J \bigcup M$ and there exists $y(\neq x) \in L$ for every $u \in x^{\curlywedge}$ and for every $v \in x^{\curlyvee}$ such that $u \prec y \prec v$.*

Proof. Let L be a graded lattice and $x \in L$ be $\mathcal{G}$-deletable. By Lemma 2.1, x must be either in $J - M$ or $M - J$ or $J \bigcap M$, otherwise $L - x$ is not a lattice and consequently $x \in J \bigcup M$. If there does not exist y for some $u \in x^{\curlywedge}$ and for some $v \in x^{\curlyvee}$ such that $u \prec y \prec v$, then $r(u)$ is not well defined and consequently $L - x$ is not graded, a contradiction.

Conversely, suppose that $x \in J \bigcup M$ and there exists $y(\neq x) \in L$ for every $u \in x^{\curlywedge}$ and for every $v \in x^{\curlyvee}$ such that $u \prec y \prec v$. If $x \in J - M$ or $x \in M - J$ or $x \in J \bigcup M$, by Lemma 2.1 $L - x$ is a lattice. Now the $r(z), \forall z \in L - x$ is same as $r(z), \forall z \in L$, hence x is $\mathcal{G}$-deletable. $\square$

Remark 2.12. *Every element other than 0 and 1 of a Boolean lattice is deletable and accordingly we have the following Theorem and remark.*

Theorem 2.13. *Let $\mathbf{L} = \mathbf{2^n}$, $n \geqslant 2$ then $red(\mathbf{L}, \mathcal{G}) = 1$.*

Remark 2.14. *If $\mathbf{L}$ is a graded lattice, then $red(\mathbf{L}, \mathcal{G}) > 1$.*

Definition 2.15. *A finite lattice is called planar if its diagram can be drawn in the plane with non-intersecting lines.*

Theorem 2.16. *Let L be a planar lattice and $x(\neq 0,1) \in L$. Then x is $\mathcal{PL}$-deletable if and only if $x \in J \bigcup M$.*

Proof. Let L be a planar lattice and $x \in L$ be $\mathcal{CO}$-deletable. By Lemma 2.1, x must be either in $J - M$ or $M - J$ or $J \bigcap M$, otherwise $L - x$ is not a lattice and consequently $x \in J \bigcup M$.

Conversely, suppose that $x \in J \bigcup M$. If $x \in J - M$ or $x \in M - J$ or $x \in J \bigcup M$, by Lemma 2.1, $L - x$ is a lattice, hence $L - x$ is a planar lattice. $\square$

Remark 2.17. *The class of planar lattices is reducible and so, if $\mathbf{L}$ is a planar lattice, then $red(\mathbf{L}, \mathcal{P}L) = 1$.*

It is also khown that the class of strong, dually strong, lower (respectively, upper) balanced and join (respectively, meet) semidistributive lattices are reducible. Hence we have the following.

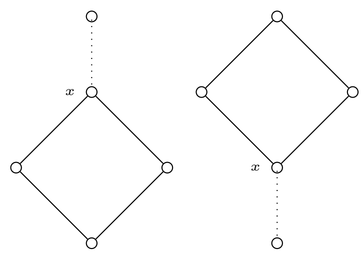
Remark 2.18. *If L is a strong lattice, then $red(\mathbf{L}, \mathcal{S}) = 1$.*
If L is a dually strong lattice, then $red(\mathbf{L}, \mathcal{D}S) = 1$.
If L is a lower balanced lattice, then $red(\mathbf{L}, \mathcal{L}B) = 1$.
If L is a upper balanced lattice, then $red(\mathbf{L}, \mathcal{U}B) = 1$.
If L is a join semidistributive lattice, then $red(\mathbf{L}, \mathcal{J}SD) = 1$.
If L is a meet semidistributive lattice, then $red(\mathbf{L}, \mathcal{M}SD) = 1$.

Also, the class of balanced, atomistic, AC, semidistributive, complemented, uniquely complemented and relatively pseudocomplemented lattices are not reducible. Hence we have the following.

Remark 2.19. *If L is a balanced lattice, then $red(\mathbf{L}, \mathcal{B}) > 1$.*
If L is a atomistic lattice, then $red(\mathbf{L}, \mathcal{A}) > 1$.
If L is a AC lattice, then $red(\mathbf{L}, \mathcal{AC}) > 1$.
If L is a semidistributive lattice, then $red(\mathbf{L}, \mathcal{S}D) > .1$
If L is a complemented lattice, then $red(\mathbf{L}, \mathcal{C}) > 1$
If L is a uniquely complemented lattice, then $red(\mathbf{L}, \mathcal{U}C) > 1$.
If L is a relatively pseudocomplemented lattice, then $red(\mathbf{L}, \mathcal{R}SC) > 1$.

3. INFINITE LATTICES

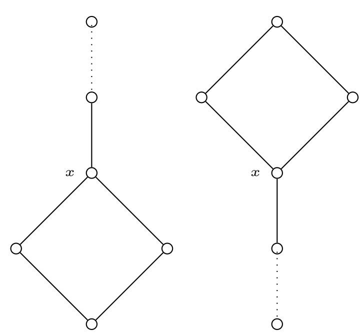
Consider the lattices given below and we observe that if we delete x from either lattice then the resultant poset is not a lattice.



(a) (b)

Theorem 3.1. *Let L be a lattice and $x(\neq 0,1) \in L$. Then $L - x$ is a lattice if and only if x satisfies one of the following conditions.*

(i) $x \in J \bigcap M$
(ii) $x \in J - M$ *and* $x^{\curlywedge} \neq \emptyset$
(iii) $x \in M - J$ *and* $x^{\curlyvee} \neq \emptyset$



(a) (b)

Proof. Let L be a lattice and $x(\neq 0,1) \in L$. Suppose that $L - x$ is a lattice and x does not satisfy any of the conditions stated in the statement. Then there exist distinct elements x_1, x_2, x_3, x_4 distinct from x in L with $x_1 \parallel x_2$ and $x_3 \parallel x_4$ such that $x_1 \vee x_2 = x = x_3 \wedge x_4$ in L. Now, $x_1, x_2 \leqslant x_1 \vee x_2 = x_3 \wedge x_4 \leqslant x_3, x_4$ in $L - x$, and we have, $x_1 \leqslant x_3, x_1 \leqslant x_4$ and

$x_2 \leqslant x_3, x_2 \leqslant x_4$ in $L - x$, i.e, x_1, x_2 are the lower bounds of x_3, x_4 and x_3, x_4 are the upper bounds of x_1, x_2 in L. Since $x_1 \parallel x_2$, suppose w is the greatest lower bound for x_3, x_4. Therefore $w \leqslant x_3, x_4$ but in L, $x = x_3 \wedge x_4$. Thus in L, $x > w$ and $x = x_1 \vee x_2$. However, $L - x$, w is an upper bound for x_1, x_2 implies that in L, $x \leqslant w$, and a contradiction. Hence there does not exist the greatest lower bound for x_3, x_4 and similarly $x_3 \parallel x_4$ implies that there does not exist least upper bound for x_1, x_2. Thus $L - x$ is not a lattice. Therefore x must satisfy one of the conditions in the statement.

Conversely, suppose that $x \in L$ satisfies one of the conditions (i), (ii) and (iii).

(i) $\underline{x \in J \cap M}$:
In this case, x is not a join as well as not a meet of any two elements in L. For $x_1 (\neq x), x_2 (\neq x) \in L$, if $x_1 \wedge x_2 = x$ in L, then $x_1 \wedge x_2 = x^-$ in $L - x$ and if $x_1 \vee x_2 = x$ in L, then $x_1 \vee x_2 = x^+$ in $L - x$. All other meets and joins of any two elements in L are preserved in $L - x$ and so $L - x$ is a lattice.

(ii) $\underline{x \in J - M \text{ and } x^{\curlywedge} \neq \emptyset}$:
Since $x \notin M$, there exist $x_1 (\neq x), x_2 (\neq x) \in L$ with $x_1 \wedge x_2 = x$ and $x^- \in L$ with $x^- \prec x$. Thus in $L - x$, $x_1 \wedge x_2 = x^-$ and so the meet of any two elements exists. Also, x is not a join of any two elements, thus join of any two elements in L is preserved in $L - x$. Hence $L - x$ is a lattice.

(iii) $\underline{x \in M - J \text{ and } x^{\curlyvee} \neq \emptyset}$:
Since $x \notin J$, there exist $x_1 (\neq x), x_2 (\neq x) \in L$ with $x_1 \vee x_2 = x$ in L and $x^+ \in L$ with $x \prec x^+$. Therefore in $L - x$, $x_1 \vee x_2 = x^+$, hence $x_1 \vee x_2$ exists in $L - x$. Also, meet of any two elements in L is preserved in $L - x$. Thus $L - x$ is a lattice.

Hence in each case, $L - x$ forms a lattice. □

Definition 3.2 (Grätzer, 1998). *A partially ordered set $(L, \leqslant)$ is a complete lattice if every subset A of L has both a greatest lower bound (the infimum, also called the meet) and a least upper bound (the supremum, also called the join) in $(L, \leqslant)$.*

Definition 3.3 (Grätzer, 1998). *Let L be a complete lattice and let a be an element of L. Then a is called compact if $a \leqslant \bigvee X$, for any $X \subseteq L$, implies that $a \leqslant \bigvee X_1$ for some finite $X_1 \subseteq X$. A complete lattice is called algebraic if every element is the join of a (possibly infinite) set of compact elements.*

In the next Theorem, we provide a necessary and sufficient condition for deletable elements of algebraic lattices.

Theorem 3.4. *Let L be an algebraic lattice and $x (\neq 0, 1) \in L$. Then x is $\mathcal{AL}$-deletable if and only if x satisfies one of the following conditions.*

(i) $x (\neq 1) \in M - J$ with $x^{\curlyvee} \neq \emptyset$.
(ii) $x (\neq 0) \in J - M$ with $x^{\curlywedge} \neq \emptyset$ and $y \notin J(L - x)$ for any $y \in x^{\curlyvee}$.
(iii) $x \in J \cap M$ and $y \notin J(L - x)$ for any $y \in x^{\curlyvee}$.

Proof. Suppose that L is an algebraic lattice and an element $x (\neq 0, 1) \in L$ is $\mathcal{AL}$-deletable. We contend that x satisfies either of the conditions stated in the statement of the Theorem. On the contrary, suppose that x does not satisfy any condition. Then by Lemma 2.1, x is doubly reducible and consequently, $L - x$ is not a lattice or equivalently x is not $\mathcal{AL}$-deletable, a contradiction to the assumption.

Conversely, suppose that x satisfies one of the conditions in the statement of the Theorem. Now, we will prove that x is $\mathcal{AL}$-deletable.

(i) $\underline{x \in M - J \text{ with } x^{\curlyvee} \neq \emptyset}$
Since $x \notin J$, there exist $x_1 (\neq x), x_2 (\neq x) \in L$ with $x_1 \vee x_2 = x$ and $x \prec x^+$ in L. Therefore in $L - x$, $x_1 \vee x_2 = x^+$ and also meet of any two elements in L is preserved in $L - x$. Thus

$L - x$ is a lattice and in fact, every element including x^+ is the join of compact elements contained in it and consequently, $L - x$ is algebraic.

(ii) $x \in J - M$ with $x^{\curlywedge} \neq \emptyset$ and $y \notin J(L-x)$ for any $y \in x^{\curlyvee}$
Since $x \notin M$, there exist $x_1(\neq x), x_2(\neq x) \in L$ with $x_1 \wedge x_2 = x$ and $x^- \prec x$ in L. In $L - x$, $x_1 \wedge x_2 = x^-$, $x_1 \wedge x_2$ exists in $L - x$ and the elements of the set $x^{\curlyvee}$ are not in $J(L-x)$, these are joins of compact elements in $L - x$. Hence $L - x$ is algebraic.

(iii) $x \in J \cap M$ and $y \notin J(L-x)$ for any $y \in x^{\curlyvee}$
By Case (ii), $L - x$ is an algebraic lattice.

Thus in each case, $L - x$ is an algebraic lattice. Hence x is $\mathcal{AL}$-deletable. □

Theorem 3.5. *Let L be a lattice and $x(\neq 0, 1) \in L$. Then x is $\mathcal{CO}$-deletable if and only if $x \in J \bigcup M$.*

Proof. Let L be a lattice and $x \in L$ be $\mathcal{CO}$-deletable. By Theorem 3.1, x is either in $J - M$ or $M - J$ or $J \bigcap M$, otherwise $L - x$ is not a lattice and consequently, $x \in J \bigcup M$.

Conversely, suppose that $x \in J \bigcup M$. If $x \in J - M$ or $x \in M - J$ or $x \in J \bigcap M$, by Lemma 2.1 $L - x$ is a lattice. □

Remark 3.6. *The class of complete lattices is reducible.*

As such, we have the following remark about the reducibility number of complete lattices.

Remark 3.7. *If L is a Lattice, then $red(\boldsymbol{L}, \mathcal{CO}) = 1$.*

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